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A Note on Green's Function for Microstrip

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***Abstract*—The electrostatic Green's function for the open or covered microstrip line is obtained by an integral representation of the free space Green's function, the results of which may be applied to obtain approximately the characteristics of the lowest order "quasi-TEM" mode of microstrip.**

I. INTRODUCTION

The electrostatic Green's function for the open microstrip line may be obtained from extended image theory as illustrated by Silvester [1] and later by Weeks [2]. Weiss and Bryant [3] derived the covered microstrip Green's function by using a computer algorithm and this was refined by Farrar and Adams [4]. However, in [4] the Green's function is not explicitly given for all values of b/h[1] and the logarithmic singularity inherent in their series representation is not immediately apparent. A direct method of obtaining the Green's function for an open or covered microstrip without using extended image theory or a computer algorithm, valid for any b/h and retaining the logarithmic singularity, is now given. The approach used is conceptually similar to that of Kaden [5], in which the capacitance of a pair of circular cylindrical wires above a dielectric coated ground plane was determined.

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[1] b is the separation of the ground planes and h is the height of the line source from the bottom ground plane.

II. FREE-SPACE GREEN'S FUNCTION

The free-space Green's function satisfying

$$\nabla^2 \phi_0 = -\delta(x,y) \tag{1}$$

is [6]

$$\phi_0 = \frac{-1}{2\pi\epsilon_0} \log (r) \tag{2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \qquad \delta(x,y)$$

is the two-dimensional Dirac delta function, ϵ_0 is the permittivity of free space, and $r = (x^2 + y^2)^{1/2}$. An integral representation of $\log (z)$ with $z = y + jx$ and $j^2 = -1$ is [7]

$$\log (z) = \int_0^\infty \frac{\exp(-\lambda) - \exp(-\lambda z)}{\lambda} \, d\lambda \tag{3}$$

where the integral converges provided $\operatorname{Re}(z) \geq 0$. On taking the real part of (3) and substituting for $\log (r)$ in (2), the free-space Green's function takes the form

$$\phi_0 = \frac{1}{2\pi\epsilon_0} \int_0^\infty \frac{\exp(-\lambda |y|) \cos(\lambda x) - \exp(-\lambda)}{\lambda} \, d\lambda. \tag{4}$$

Note that in (4) the x and y variables are separated so that derivatives may be easily obtained. Now the integral representation (4) is used to derive the electrostatic Green's function, the results of which may be applied to obtain approximately the characteristics of the lowest order "quasi-TEM" mode of a microstrip.

III. OPEN MICROSTRIP GREEN'S FUNCTION

To ϕ_0 are added functions satisfying the two-dimensional Laplace's equation, exhibiting the same x behavior as (4) and together with ϕ_0 satisfying the appropriate boundary conditions. Thus with reference to Fig. 1, the Green's function for the open microstrip may be chosen as

$$\phi_1(x,y) = \frac{1}{2\pi\epsilon_0} \cdot \int_0^\infty \frac{\exp(-\lambda |y - \Delta|) + f_1(\lambda) \exp[-\lambda(y - \Delta)]}{\lambda} \cdot \cos(\lambda x) \, d\lambda \tag{5}$$

for $y > 0$, and

$$\phi_2(x,y) = \frac{1}{2\pi\epsilon_0\epsilon_r} \cdot \int_0^\infty \frac{f_2(\lambda) \exp[-\lambda(y - \Delta)] + f_3(\lambda) \exp[\lambda(y - \Delta)]}{\lambda} \cdot \cos(\lambda x) \, d\lambda \tag{6}$$

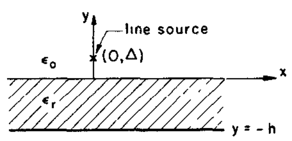

Fig. 1. Green's function geometry for the open microstrip.

for $y < 0$. Here Δ is defined in Fig. 1, ϵ_r is the dielectric constant, and the functions $f_i(\lambda)$, $i = 1,2,3$ are to be determined from the prevailing boundary conditions. The convergence of the integrals (5) and (6) may be verified after the $f_i(\lambda)$ are found. The boundary conditions stated below and the representations (5), (6), yield the following relations between the $f_i(\lambda)$:

$$\phi_1(x,0) = \phi_2(x,0)$$

$$\exp(-\lambda|\Delta|) + f_1(\lambda)\exp(\lambda\Delta) = \frac{1}{\epsilon_r}\{f_2(\lambda)\exp(\lambda\Delta) + f_3(\lambda)\exp(-\lambda\Delta)\} \quad (7a)$$

$$\frac{\partial\phi_1(x,0)}{\partial y} = \epsilon_r\frac{\partial\phi_2(x,0)}{\partial y}$$

$$\exp(-\lambda|\Delta|) - f_1(\lambda)\exp(\lambda\Delta) = -f_2(\lambda)\exp(\lambda\Delta) + f_3(\lambda)\exp(-\lambda\Delta) \quad (7b)$$

$$\phi_2(x,-h) = 0$$

$$f_2(\lambda)\exp[\lambda(h+\Delta)] + f_3(\lambda)\exp[-\lambda(h+\Delta)] = 0 \quad (7c)$$

(7a), (7b), (7c) are 3 equations for the unknown $f_i(\lambda)$. Since $\phi_1 = \phi_2$ at $y = 0$ there is no need to evaluate (5), so that f_1 is not required. The limit $\Delta \to 0$ in (7) gives

$$f_2(\lambda) = \frac{-A\exp(-2\lambda h)}{1 - K\exp(-2\lambda h)} \quad (8a)$$

$$f_3(\lambda) = \frac{A}{1 - K\exp(-2\lambda h)} \quad (8b)$$

where $K = (1-\epsilon_r)/(1+\epsilon_r)$ and $A = 2\epsilon_r/(1+\epsilon_r)$. For these f_2,f_3 the integral (6) is convergent. Since $|K\exp(-2\lambda h)| < 1$, (8a) and (8b) are readily expanded into uniformly convergent series

$$f_2(\lambda) = -A\sum_{n=0}^{\infty} K^n \exp[-2\lambda h(n+1)] \quad (9a)$$

$$f_3(\lambda) = A\sum_{n=0}^{\infty} K^n \exp(-2\lambda hn). \quad (9b)$$

Substitute (9) in (6) and interchange the order of summation and integration; the Green's function for the open microstrip line with

$$\phi_2^*(x,y) = \pi\epsilon_0(1+\epsilon_r)\phi_2(x,y)$$

becomes

$$\phi_2^*(x,y) = \sum_{n=0}^{\infty} K^n \int_0^\infty \frac{\exp(-\lambda y_{n1}) - \exp(-\lambda y_{n2})}{\lambda}\cos(\lambda x)\,d\lambda \quad (10)$$

where $y_{n1} = -y + 2hn$ and $y_{n2} = y + 2h(n+1)$. Note that (10) may be evaluated by inspection; it is evident from the definiton (4) that

$$\int_0^\infty \frac{\exp(-\lambda y_{n1}) - \exp(-\lambda y_{n2})}{\lambda}\cos(\lambda x)\,d\lambda = \log\left(\frac{r_{n2}}{r_{n1}}\right) \quad (11)$$

where $r_{n1} = (y_{n1}^2 + x^2)^{1/2}$ and $r_{n2} = (y_{n2}^2 + x^2)^{1/2}$. Equation (11) applied to (10) gives the open microstrip Green's function, for the origin (ξ,η) as

$$\phi_2(x,y,\xi,\eta) = \frac{1}{2\pi\epsilon_0(1+\epsilon_r)} \cdot \sum_{n=0}^{\infty} K^n \log\frac{[2h(n+1) + (y-\eta)]^2 + [x-\xi]^2}{[2hn - (y-\eta)]^2 + [x-\xi]^2}. \quad (12)$$

Silvester and Benedek obtained this result in [8] by extended image theory; Weeks [2] obtained the Green's function for a more general location of the source point, again by using image theory.

IV. COVERED MICROSTRIP GREEN'S FUNCTION

With reference to Fig. 2, the Green's function for the covered microstrip may be chosen as

$$\psi_1(x,y) = \frac{1}{2\pi\epsilon_0}\int_0^\infty \frac{\exp(-\lambda|y-\Delta|) + f_1(\lambda)\exp[-\lambda(y-\Delta)] + f_2(\lambda)\exp[\lambda(y-\Delta)]}{\lambda}\cos(\lambda x)\,d\lambda \quad (13)$$

for $y > 0$, and

$$\psi_2(x,y) = \frac{1}{2\pi\epsilon_0\epsilon_r}\int_0^\infty \frac{f_3(\lambda)\exp[-\lambda(y-\Delta)] + f_4(\lambda)\exp[\lambda(y-\Delta)]}{\lambda}\cos(\lambda x)\,d\lambda \quad (14)$$

for $y < 0$. The functions $f_i(\lambda)$, $i = 1,2,3,4$ are determined from the 4 boundary conditions,

$$\psi_1(x,b) = 0,$$
$$\psi_2(x,-h) = 0,$$
$$\psi_1(x,0) = \psi_2(x,0),$$
$$\frac{\partial\psi_1(x,0)}{\partial y} = \epsilon_r\frac{\partial\psi_2(x,0)}{\partial y}. \quad (15)$$

Using the same procedure as in Section III, the Green's function for the covered microstrip is

$$\psi_2(x,y,\xi,\eta) = \frac{1}{2\pi\epsilon_0(1+\epsilon_r)}\sum_{n=0}^{\infty} C_n \log\frac{T_1T_3}{T_2T_4} \quad (16)$$

where

$$C_n = \frac{n!}{n_1!n_2!n_3!}(-)^{n_1}K^{n_1+n_2}, \qquad n = n_1 + n_2 + n_3$$

$K = (1-\epsilon_r)/(1+\epsilon_r)$, and T_k, $k = 1,2,3,4$ are given by

$$T_1 = \{2b(n_1+n_3) + 2h(1+n_2+n_3) + (y-\eta)\}^2 + \{x-\xi\}^2$$
$$T_2 = \{2b(1+n_1+n_3) + 2h(1+n_2+n_3) + (y-\eta)\}^2 + \{x-\xi\}^2$$
$$T_3 = \{2b(1+n_1+n_3) + 2h(n_2+n_3) - (y-\eta)\}^2 + \{x-\xi\}^2$$
$$T_4 = \{2b(n_1+n_3) + 2h(n_2+n_3) - (y-\eta)\}^2 + \{x-\xi\}^2.$$

Note that the summation employed in (16) is to be performed for all combinations of (n_1,n_2,n_3) that give $n = 0,1,2,\cdots$ (e.g., for $n = 2$ there are 6 combinations of n_1,n_2,n_3).

The covered microstrip Green's function (16), which is apparently new, contains the logarithmic singularity corresponding to $n_1 = n_2 =$

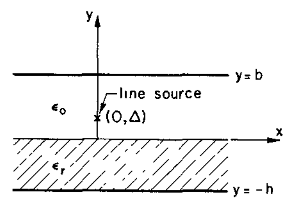

Fig. 2. Green's function geometry for the covered microstrip.

$n_3 = 0$ in T_4, so that (16) may be written as

$$\psi_2(x,\xi,y,\eta) = \frac{-1}{2\pi\epsilon_0(1+\epsilon_r)} \log \{(x-\xi)^2 + (y-\eta)^2\} + H(x,\xi,y,\eta) \tag{17}$$

where $H(x,\xi,y,\eta)$ is continuous. For computational purposes this separation of the singularity is important.

For $b \to \infty$ (16) reduces to the open microstrip Green's function (12) whereas for $\epsilon_r = 1$ the only nonzero terms in (16) occur with $n_1 = n_2 = 0$ and it corresponds to the *in vacuo* case of a microstrip between ground planes.

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Field Propagation in an Open-Beam Waveguide

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***Abstract*—A new method is described for the investigation of the open-beam waveguides, which may be applied even when the optical elements are not inserted in absorbing screens.**

The theory of an open-beam waveguide deals, in general, with the determination of the iterative beams associated with it [1]. When the finite size of the optical elements composing the beam waveguide is taken into account, the iterative beams turn out to be strictly related to the oscillation modes of a suitable defined two-mirror open resonator, equivalent to the beam waveguide [2], [3]. Such an open resonator is composed of mirrors which behave in reflection like the optical elements of the waveguide behave in transmission.

An alternative method of treating the open-beam waveguides is to study how the field originates from a given source, at a given distance from the first guiding element, propagating from cell to cell. To this end, we can choose a plane per cell, for example, at the input of each guiding element, and evaluate the field at the points of the plane π_{n+1} in terms of the field at the points of the plane π_n (Fig. 1). By referring to a two-dimensional problem, where the quantities of interest depend only on the longitudinal coordinate z and on one transverse coordinate x, we can write, in the scalar approximation,

$$v_{n+1}(x_{n+1}) = \int_{-\infty}^{\infty} v_n(x_n) K_{n,n+1}(x_n,x_{n+1}) dx_n \tag{1}$$

where v_i denotes the field (impinging) at the points of the plane π_i, x_i,z_i, the coordinates of the points of π_i, and $K_{i,i+1}$, the Green's function describing the propagation from the plane π_i to the plane π_{i+1}, including the transmission through L_i.

Starting with the source distribution $v_0(x_0)$, (1) allows us to evaluate $v_1(x_1)$, then $v_2(x_2)$, and so on, in other words, to study the evolution of the field through the waveguide. In the general case, the evaluation of the fields $v_i(x_i)$ must be done numerically, using an electronic computer. This implies some practical difficulty when the guiding elements are not inserted in absorbing screens. If the guiding elements are "diaphragmed" (Fig. 2), (1) can be written in the form

$$v_{n+1}(x_{n+1}) = \int_{a_n}^{b_n} v_n(x_n) K_{n,n+1}(x_n,x_{n+1}) dx_n \tag{2}$$

where $b_n - a_n$ represents the aperture of the nth pupil, and the evaluation of v_{n+1} requires an integration over a finite interval, which is easily done by the electronic computers. When the pupil apertures have infinite width (only partially occupied by the guiding elements, in practical cases), the integration is to be made over an infinite interval, which creates problems of accuracy of the results.

It occurred to us, however, that this difficulty can be overcome in the following way. Let us denote by ξ_n and ξ_n' the limits of the region of the plane π_n occupied by L_n. Equation (1) may be rewritten in the form

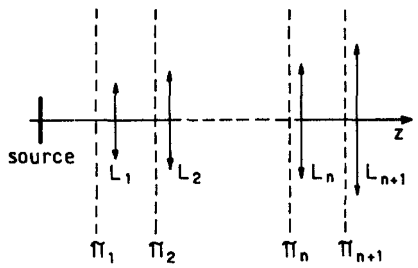

Fig. 1. A general beam waveguide composed of the elements L_i, with the reference planes π_i.

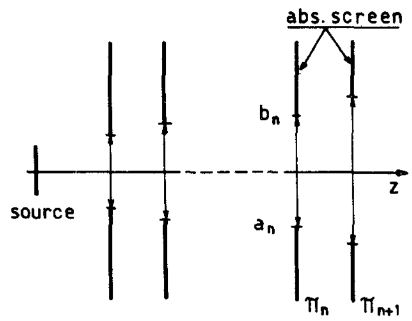

Fig. 2. A sequence of diaphragmed guiding elements.

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